

NONNEGATIVELY CURVED SUBMANIFOLDS IN CODIMENSION TWO

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ABSTRACT. Let M be a complete noncompact manifold with nonnegative sectional curvatures isometrically immersed in Euclidean spaces with codimension two. We show that M is a product over its soul, except when the soul is the circle S^1 or M is 3-dimensional and the soul is the Real Projective Plane. We also give a rather complete description of the immersion, including the exceptional cases.

1. INTRODUCTION

Let $f: M^n \rightarrow \mathbb{R}^{n+2}$ be an isometric immersion of a complete Riemannian manifold M with nonnegative sectional curvatures ($k \geq 0$). If M is compact, we have a topological classification of this manifold in [B-M₁ and B-M₂]. In this paper we want to consider the noncompact case. In [C-G], Cheeger and Gromoll have shown that a complete manifold with $k \geq 0$ is diffeomorphic to the total space of a vector bundle over a compact submanifold, its soul, and classified it in dimensions ≤ 3 , up to isometry. Then it is an interesting problem to know under which conditions M turns out to be a product over its soul. In this paper show that this is always the case, except when the soul is the circle S^1 or the Real Projective Plane $\mathbb{R}P^2$. This is the content of Theorem 1. We also give a rather complete description of the immersion, including these exceptional cases (Theorem 1(i) and Theorem 2). In a previous paper [N₁], we have some topological results. In another paper, [N₂], we prove that if there is a point $p \in M$ such that all the sectional curvatures are positive, then the soul of M is a point, implying that M is diffeomorphic to the Euclidean space \mathbb{R}^n .

Before stating our results, we recall that f is called an l -cylindrical immersion if there exist the factorizations $M^n = M_1^{n-l} \times \mathbb{R}^l$ and $f = f_1 \times i$, where $i: \mathbb{R}^l \rightarrow \mathbb{R}^l$ is the identity map. Then, in order to classify M and f , we can suppose that f is noncylindrical which is equivalent by Hartman [H] to the existence of a point x in M such that the index of relativity nullity $\nu(x) = 0$, where

$$\nu(x) = \dim\{X \in T_x M \mid \alpha(X, Y) = 0, \forall Y \in T_x M\}$$

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and α is the second fundamental form. In what follows, f will always be a noncylindrical immersion. Our first result states

Theorem 1. *Let $f: M^n \rightarrow \mathbb{R}^{n+2}$, $n \geq 3$, be a noncylindrical isometric immersion of a complete noncompact manifold with nonnegative sectional curvatures and with an m -dimensional soul A^m . Then we have*

(i) *If M is simply connected then M is isometric to the product of a manifold homeomorphic to the sphere S^m with a complete manifold \mathcal{P}^{n-m} diffeomorphic to \mathbb{R}^{n-m} . If $n - m > 1$, f is a product of a hypersurface immersions.*

(ii) *If M is not simply connected, the soul must be homeomorphic to the circle S^1 or to the Real Projective Space $\mathbb{R}P^2$. The latter case occurs only when $n = 3$.*

It follows by the theorem above that when M is not simply connected, M can be classified topologically. M will be diffeomorphic:

(i) either to $S^1 \times \mathbb{R}^{n-1}$ or to the total space of a nonorientable vector bundle over S^1 .

(ii) either to $\mathbb{R}P^2 \times \mathbb{R}$ or to total space of a nonorientable line bundle over $\mathbb{R}P^2$.

It also follows by Theorem 1 and the classification of compact manifolds immersed in codimension two that the immersion will be cylindrical over its soul when the soul is homeomorphic to a flat torus or a two-dimensional Klein bottle, or a product of two spheres, or the total space of a nonorientable fiber bundle over S^1 .

When the codimension of the soul is 1, we cannot expect to have a product of hypersurfaces in the Theorem 1, as we see in the following example: let $g: M_1^{n_1} \rightarrow \mathbb{R}^{n_1+1}$ be a codimension one isometric immersion and consider

$$M_1 \times \mathbb{R} \xrightarrow{g \times i} \mathbb{R}^{n_1+2} \xrightarrow{h} \mathbb{R}^{n_1+3}$$

where h is an isometric immersion. $f = h \circ (g \times i)$ is not in general either a product of immersions nor 1-cylindrical. For this case we prove

Theorem 2. *Let $f: M^n \rightarrow \mathbb{R}^{n+2}$ be a noncylindrical isometric immersion, where $M = M_1^{n-1} \times \mathbb{R}$ is a simply connected manifold with nonnegative sectional curvatures and $n \geq 3$. Then*

(i) *There exists an isometric immersion g from M^{n-1} to \mathbb{R}^n .*

(ii) *Suppose that for each $x \in M$, there is a two-plane $\sigma \subset T_x M$ such that $k(\sigma) > 0$; then f is homotopic through isometric immersions to the immersion $g \times i: M = M_1 \times \mathbb{R} \rightarrow \mathbb{R}^{n+1}$, where i is the identity map.*

Theorem 2 also covers the case of the universal covering of the manifolds which appear in Theorem 1(ii). We have not been able to construct a proof for Theorem 2(ii) without an extra condition for $k(\sigma)$. But we want to observe that the same statement can be proved if we do not require differentiability of the isometric homotopy.

Finally, by adding one extrinsic property to the immersion we are able to show that f is always a product:

Theorem 3. *With the same hypothesis of Theorem 1, suppose that the immersion has flat normal bundle ($\mathcal{R}^\perp = 0$). Then M is a product over the soul A and f is a product of hypersurface immersions. It follows that the soul is homeomorphic to a sphere S^m , $m \geq 1$.*

At the end of §5 in this paper, modifying slightly the Yeaton Clifford example which appears in [A-M], we see that we can have a complete noncompact manifold isometrically immersed in codimension two, which is a product over the soul S^1 but the immersion f is not a product of hypersurface immersions.

In order to have a complete understanding of isometric immersions of nonnegatively curved manifolds in \mathbb{R}^{n+2} , it seems reasonable to ask if a manifold homeomorphic to $\mathbb{R}P^2$ with $k \geq 0$ can be isometrically immersed in \mathbb{R}^N with the dimension of the first normal space at most two, since Gromov and Rokhlin in [G-R, p. 42] have proved that if $k > 0$, $\mathbb{R}P^2$ cannot be embedded in \mathbb{R}^4 .

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2. PRELIMINARIES AND REDUCIBILITY ALONG THE SOUL

It is a well-known result of Weinstein [W], that if the codimension of an isometric immersion is two then the nonnegativity of the sectional curvatures ($k \geq 0$) implies the nonnegativity of the curvature operator ($\mathcal{R} \geq 0$).

For the case where M^n is a complete noncompact manifold with $\mathcal{R} \geq 0$, we now collect some known results of a soul A of M .

(2.1) A soul A of M is a compact, totally geodesic submanifold of M , without boundary and has $\mathcal{R} \geq 0$ (see [C-G]).

(2.2) If M is simply connected, M is isometric to the product $A^m \times \mathcal{P}^{n-m}$ where \mathcal{P}^{n-m} is a complete manifold diffeomorphic to \mathbb{R}^{n-m} (see [N₂]).

(2.3) If M is not simply connected, M is locally isometric to a product over the soul A (see [N₂]).

In order to prove our theorems, we need the following concept.

(2.4) **Definition.** An isometric immersion $f: M^n \rightarrow \mathbb{R}^{n+p}$, $p \geq 1$, is called *reducible along A* if for all $X \in TA$ and $Y \in TA^\perp$ we have $\alpha(X, Y) = 0$.

As it is proved in [B-N], this property implies that M is diffeomorphic to $A^m \times \mathbb{R}^{n-m}$. Another consequence of the reducibility along A is given by the following proposition.

(2.5) **Proposition.** Let $f: M^n \rightarrow \mathbb{R}^{n+p}$, $p \geq 1$, be an isometric immersion of a complete noncompact manifold with nonnegative curvature operator. If f is reducible along a soul A then M is isometric to a product $A^k \times \mathcal{P}^{n-k}$.

Proof. By (2.3) there is a Riemannian submersion $P: M \rightarrow A$ where the fibers, F are totally geodesic. Hence it follows from [O] that to show that M splits as a product, it is sufficient to prove that for each geodesic loop $\gamma: [0, 1] \rightarrow A$, starting from a point $x \in A$, the map

$$L_\gamma: \pi^{-1}(x) = F_x \rightarrow F_x, \quad L_\gamma(p) = \tilde{\gamma}(1),$$

(where $\tilde{\gamma}$ denotes the horizontal lift of γ with $\tilde{\gamma}(0) = p$) is the identity on F_x .

This will follow easily from the fact that $\tilde{\gamma}(0) = p$ belongs to the pseudo-soul \bar{A} (see [Y]) obtained in the following way:

Consider $\sigma: [0, l] \rightarrow M$ the minimal unit speed geodesic connecting p to the soul A . The vector $Y_{\sigma(l)} = -\sigma'(l)$ is a normal vector of A . Since f is

reducible along A , we have

$$\tilde{\nabla}_X Y = \nabla_X Y + \alpha(X, Y) = \nabla_X Y \quad \text{for all } X \in TA$$

(where $\tilde{\nabla}$ and ∇ are the Riemannian connections of \mathbb{R}^{n+p} and M respectively) implying that the parallel translation of the vector $Y_{\sigma(l)}$ along A in \mathbb{R}^{n+p} gives a normal parallel section Y along A .

Then the map $\varphi_Y: A \times \mathbb{R} \rightarrow M$, given by $\varphi_Y(x, t) = \exp_x tY(x)$ is an isometric immersion (see [Y, Proposition 3.2], $\bar{A} = \varphi_Y(A, l)$ and $\tilde{y} = \varphi_Y(\gamma, l)$).

3. BASIC LEMMA AND PROOF OF THEOREM 1

Let $r(x)$ be the Lie algebra generated by the range of the curvature operator \mathcal{R} at the point x . Let U be the orthogonal complement of the relative nullity subspace $N(x)$. Since the codimension of the immersion is 2, by Theorem 1 in [Bi] we have the following possibilities for $r(x)$:

- (a) $r(x) = \theta(U)$,
- (b) $r(x) = \theta(V_1) \oplus \theta(V_2)$ where $V_1 \oplus V_2 = U$,
- (c) there is a complex structure on U and $r(x)$ is the unitary algebra of this structure (and the second fundamental forms are all of signature zero unless $\dim U = 4$),

where $\theta(U)$ is the exterior algebra of U . Moreover, if (b) occurs with $\dim V_1 > 1$ and $\dim V_2 > 1$ then V_1 and V_2 are orthogonal to each other and $R^\perp(x) = 0$ where R^\perp is the normal curvature tensor of f . (See Theorem 1(a') of [Bi].)

We will use the classification above to prove the lemma below whose hypothesis is satisfied by our three theorems stated in §1. We will use the notation A_ξ for the Weingarten operator, where ξ is a normal vector.

(3.1) Lemma. *Let $f: M^n \rightarrow \mathbb{R}^{n+2}$ be an isometric immersion. Suppose that there exists an open set O of M such that O is a metric product $O_1 \times O_2$. Then for $x \in O$ we have*

- (a) *If $\mathcal{R}|_{\theta(T_x O_i)} \neq 0$, $i = 1, 2$, there is an orthonormal frame $\{\xi_1, \xi_2\}$ in the normal space such that $A_{\xi_1}|_{T_x O_2} = 0$ and $A_{\xi_2}|_{T_x O_1} = 0$.*
- (b) *If $\mathcal{R}|_{\theta(T_x O_2)} = 0$ and $\alpha(Z, Z) \neq 0$ for some $Z \in T_x O_2$, there is an orthonormal frame $\{\xi_1, \xi_2\}$ such that $A_{\xi_1}(T_x O_1) \subseteq T_x O_1$, $A_{\xi_1}(Z) = 0$ and $\text{rank } A_{\xi_2} = 1$.*

Proof. (a) The only possibility in Bishop's classification is $r(x) = \theta(V_1) \oplus \theta(V_2)$ with $\dim V_1 > 1$ and $\dim V_2 > 1$ whence $\mathcal{R}^\perp(x) = 0$. Therefore, the conclusion follows by Bishop's Theorem.

(b) By hypothesis $\dim U \geq 1$. If $\dim U = 1$ there exists a normal vector ξ such that A_ξ is identically null and then $\xi_1 = \xi$ and ξ_2 orthogonal to ξ satisfy the lemma.

If $\dim U = 2$, we will define ξ_2 by $\alpha(Z, Z)/\|\alpha(Z, Z)\|$. Taking ξ_1 orthogonal to ξ_2 , $\langle A_{\xi_1} Z, Z \rangle = 0$. But there exists a vector Z_1 which is the eigenvector of A_{ξ_1} associated to its only nonnull eigenvalue. We claim that $Z_1 \in T_x O_1$. In fact, $Z_1 = aX + bZ$, where X is the normalized orthogonal projection of Z_1 onto $T_x O_1$. If we take $Z_1^\perp = -bX + aZ$, $A_{\xi_1}(Z_1^\perp) = 0$ then,

$$0 = -bA_{\xi_1} X + aA_{\xi_1} Z.$$

Taking inner product with Z (remembering that $\langle A_{\xi_1} Z, Z \rangle = 0$) we have

$$= 0 - b\langle A_{\xi_1} X, Z \rangle.$$

If $b = 0$ we have $Z_1 = X$ and if $\langle A_{\xi_1} X, Z \rangle = 0$ we have, because $A_{\xi_1}(Y) = 0$ for Y orthogonal to X , $Y \in T_x M_1$, that $T_x M_1$ is invariant by A_{ξ_1} and $A_{\xi_1}(Z) = 0$. Hence the lemma follows, for this case.

If $\dim U \geq 3$, it follows again by Bishop's classification that $r(x)$ is a nonnull algebra implying that $\mathcal{R}|_{\theta(T_x O_1)} \neq 0$. We claim that the only possibility for $r(x)$ is $\theta(V_1) \oplus \theta(V_2)$ where $\dim V_2 = 1$. In fact, consider an orthonormal frame $\{X_1, \dots, X_n\}$ of $T_x O$ such that $X_1, \dots, X_s \in N(x)$ and $Z \in \text{span}\{X_1, \dots, X_s, X_{s+1}\}$. We have

$$\mathcal{R}(X_i \wedge X_j) = 0, \quad i = 1, \dots, s, \quad j = 1, \dots, n.$$

Denoting by X' and X'' the orthogonal projection of the vector X onto $T_x O_1$ and $T_x O_2$ respectively, because the metric is a product, and $\mathcal{R}|_{T_x O_2} = 0$, we have

$$\mathcal{R}(Z \wedge X_j) = \mathcal{R}(Z \wedge X'_j) + \mathcal{R}(Z \wedge X''_j) = 0, \quad j = 1, \dots, n.$$

Then the range of \mathcal{R} is contained in $\theta(W)$, where $W = \text{span}\{X_{s+2}, \dots, X_n\}$ which implies

$$r(x) \subseteq \theta(n - s - 1).$$

It is obvious that $r(x)$ cannot be $\theta(U) = \theta(n - s)$. If $n - s$ is even, $r(x)$ cannot be the unitary algebra $u((n - s)/2)$, since $u((n - s)/2)$ is not contained in $\theta(n - s - 1)$. Then $r(x) = \theta(n - s - 1) \oplus \theta(1)$ as we claimed.

Then, following the proof of Bishop's theorem, there is an orthonormal frame $\{\xi_1, \xi_2\}$ of the normal space such that the range of A_{ξ_1} is V_1 and $\text{rank } A_{\xi_2} = 1$. We will need the calculation below to show that $V_1 \subseteq T_x O_1$.

Since $\text{rank } A_{\xi_2} = 1$, the Gauss equation is reduced to

$$(3.2) \quad \mathcal{R} = A_{\xi_1} \wedge A_{\xi_1}.$$

Because $\dim n \geq 3$, $r(x)$ is a nonnull algebra, implying that there exist $X, Y \in T_x M$ such that $\mathcal{R}(X \wedge Y) \neq 0$. Denoting again by X' and X'' the orthogonal projection on $T_x O_1$ and $T_x O_2$ respectively we have

$$\begin{aligned} \mathcal{R}(X \wedge Y) &= (A_{\xi_1} X)' \wedge (A_{\xi_1} Y)' + (A_{\xi_1} X)' \wedge (A_{\xi_1} Y)'' \\ &\quad + (A_{\xi_1} X)'' \wedge (A_{\xi_1} Y)' + (A_{\xi_1} X)'' \wedge (A_{\xi_1} Y)'' \end{aligned}$$

where

$$\omega_1 = (A_{\xi_1} X)'' \wedge (A_{\xi_1} Y)'' = 0$$

and

$$\omega_2 = (A_{\xi_1} X)' \wedge (A_{\xi_1} Y)'' + (A_{\xi_1} X)'' \wedge (A_{\xi_1} Y)' = 0,$$

since the range of the curvature operator is contained in $\theta(T_x O_1)$.

Taking interior product of ω_2 with $(A_{\xi_2} X)'$ we get

$$0 = i(A_{\xi_1} X)' \omega_2 = \|(A_{\xi_1} X)'\|^2 (A_{\xi_1} Y)'' - \langle (A_{\xi_1} Y)', (A_{\xi_1} X)' \rangle (A_{\xi_1} X)''$$

and therefore

$$(A_{\xi_1} Y)'' = \langle (A_{\xi_1} Y)', (A_{\xi_1} X)' \rangle \|(A_{\xi_1} X)'\|^{-2} (A_{\xi_1} X)''.$$

Taking interior product with $(A_{\xi_1} Y)'$ we get

$$\begin{aligned} 0 &= i(A_{\xi_1} Y)' \omega_2 = \langle (A_{\xi_1} X)', (A_{\xi_1} Y)' \rangle (A_{\xi_1} Y)'' - \|(A_{\xi_1} Y)'\|^2 (A_{\xi_1} X)'' \\ &= \|(A_{\xi_1} X)'\|^{-2} \{ \langle (A_{\xi_1} X)', (A_{\xi_1} Y)' \rangle^2 - \|(A_{\xi_1} X)'\|^2 \|(A_{\xi_1} Y)'\|^2 \} (A_{\xi_1} X)'' \end{aligned}$$

If $(A_{\xi_2}X)'' \neq 0$ the above relation implies $(A_{\xi_1}Y)' = \lambda(A_{\xi_1}X)'$ and then $\mathcal{R}(X \wedge Y) = (A_{\xi_1}X)' \wedge (A_{\xi_1}Y)' = 0$. Hence,

$$(3.3) \quad \text{if } \mathcal{R}(X \wedge Y) \neq 0, \text{ we have } (A_{\xi_1}X)'' = (A_{\xi_1}Y)'' = 0.$$

Consider now the orthonormal frame $\{Z_1, \dots, Z_n\}$ which diagonalizes the operator A_{ξ_2} such that $A_{\xi_2}(Z_1) = \lambda Z_1$ and $A_{\xi_2}(Z_i) = 0, i \geq 2$. Since $\mathcal{R} \neq 0$ at x , there exist Z_i, Z_j such that $\mathcal{R}(Z_i \wedge Z_j) \neq 0$. By (3.3), we have $\langle \alpha(Z_i, Z), \xi_1 \rangle = 0 = \langle \alpha(Z_i, Z), \xi_2 \rangle = 0$. This implies $\alpha(Z_i, Z) = 0$ for $i \geq 2$, as we have supposed $A_{\xi_2}(Z_i) = 0$. Since $\mathcal{R}(Z \wedge Z_i) = \mathcal{R}(Z \wedge Z_i') + \mathcal{R}(Z \wedge Z_i'') = 0$ we will have in the Gauss equation $\langle \alpha(Z_i, Z_i), \alpha(Z, Z) \rangle = 0$. Because $\alpha(Z_i, Z_i)$ is orthogonal to ξ_2 , we have $\alpha(Z, Z)$ orthogonal to ξ_1 . Now, writing the Gauss equation for the sectional curvature of a plane spanned by $X \in T_x O_1$ and $Z \in T_x O_2$ we get

$$0 = \langle A_{\xi_1}X, X \rangle \langle A_{\xi_1}Z, Z \rangle - (\langle A_{\xi_1}Z, X \rangle)^2 = -(\langle A_{\xi_1}Z, X \rangle)^2.$$

This together with (3.3) implies $A_{\xi_1}Z = 0$. Thus, range $A_{\xi_1} \subset T_x O_1$ and $A_{\xi_1}Z = 0$, as we claimed.

Remark. It follows from the proof of Lemma (3.1) that when $\mathcal{R}|_{T_x(O_1)} \neq 0$, there exists a vector $Y \in T_x O_1$ such that $\xi_1 = \alpha(Y, Y)/\|\alpha(Y, Y)\|$. This is obvious if $\mathcal{R}|_{T_x(O_2)} \neq 0$. If not, take Y belonging to $\ker A_{\xi_2}$ ($\text{rank } A_{\xi_2} = 1$) and to $T_x O_1$. There exists such Y such that $\alpha(Y, Y) \neq 0$ because $\mathcal{R}|_{T_x(O_1)} \neq 0$. Then, $\alpha(Y, Z) = 0$ because $\langle \alpha(Y, Z), \xi_1 \rangle = \langle A_{\xi_1}Z, Y \rangle = 0$ and $\langle \alpha(Y, Z), \xi_2 \rangle = \langle A_{\xi_2}Y, Z \rangle = 0$. Since the sectional curvature $K(Y, Z) = 0$, by the Gauss equation, $\alpha(Y, Y)$ is orthogonal to $\alpha(Z, Z)$. Since ξ_1 is orthogonal to $\alpha(Z, Z)$, ξ_1 can be taken as $\alpha(Y, Y)$ normalized.

Now, we start the proof of Theorem 1. Let x be a point in M such that $\nu(x) = 0$. By (2.3) M is locally a product over the soul and then we can apply Lemma (3.1) where $\dim U = n, T_x O_1 = T_x A$ and $T_x O_2 = T_x^\perp A$. It follows that the sectional curvatures along planes tangent to A are given by the eigenvalues of the operator A_{ξ_1} which is nonsingular on $T_x A$, since $\dim U = n$.

Let $\bar{f} = f|_A : A \rightarrow \mathbb{R}^{n+2}$ be the isometric immersion f restricted to the soul. Since A is a totally geodesic submanifold of M , the first normal space of \bar{f} , namely, the subspace of the normal space where the second fundamental form lies (see [E, p. 334] for definition), is at most two dimensional. We can easily generalize to \bar{f} , Theorems (2.2) and (2.3) of [B-M₁], since they depend only on the fact that the second fundamental form lies in a two-dimensional plane. The Lemma (3.1) implies that there is a point $p \in A$ such that all the sectional curvatures along planes tangent to A at p are positive. Hence [B-M₁] implies, if $m \geq 3$, where m is the dimension of the soul, that A and consequently M , are simply connected and A is homeomorphic to a sphere S^m . Also, if $m = 2$, A has to be a homeomorphic either to a sphere S^2 or to a Real Projective Space $\mathbb{R}P^2$.

If M is simply connected, then by (2.2) we have that M is a product $A^m \times \mathcal{P}^{n-m}$. If $n - m > 1$, it follows by [M₁] that either f is a product of hypersurface immersions or f takes a complete geodesic into a straight line. In the latter case, this geodesic is a line (in the sense that each segment realizes the distance between its endpoints), hence it must split off isometrically (cf.

[T]). This would imply that f is at least 1-cylindrical which contradicts our assumption. Then f is a product of hypersurface immersions.

The only thing we need to prove now is that if $n \geq 4$, the soul A cannot be homeomorphic to $\mathbb{R}P^2$. Suppose it is and consider the universal covering \tilde{M} of M . Let us consider \tilde{A} , the soul of \tilde{M} , which is homeomorphic to S^2 and the immersion $\tilde{f} = P \circ f$ where P is the covering map. Since $n - 2 \geq 2$ by [M₁], \tilde{f} will be a product of hypersurface immersions and then if $\tilde{X} \in TS^2$ and $\tilde{Y} \in (TS^2)^\perp$, $\tilde{\alpha}(\tilde{X}, \tilde{Y}) = 0$ ($\tilde{\alpha}$ is the second fundamental form of \tilde{f}). This implies that f is reducible along A . By Proposition (2.5) M is a product over the soul and again because $n - 2 \geq 2$, we would have A , homeomorphic to $\mathbb{R}P^2$, isometrically immersed in codimension 1, which is the required contradiction.

4. PROOF OF THEOREM 2

To prove this theorem, we will use the Fundamental Existence and Uniqueness Theorem for submanifolds (see [C, p. 48]). First, we will prove (ii) of the theorem and then (i) will follow easily using the same arguments. Following Moore's argument in [M₂], for each $x \in M$, we will take a neighborhood $U \subset M$ of x and on the tube $E = U \times \mathbb{R}^2$ contained in the normal bundle, consider differentiable sections ξ_1, ξ_2 such that if we define a second fundamental form $\alpha_t: TU \times TU \rightarrow E$ by

$$\alpha_t(X, Y) = \langle \alpha(X, Y), \xi_1 \rangle \xi_1 + t \langle \alpha(X, Y), \xi_2 \rangle \xi_2$$

and a normal connection $\nabla_t^\perp: TU \times E \rightarrow E$ by $\nabla_t^\perp \xi_\alpha = t \nabla^\perp \xi_\alpha$ for $\alpha = 1, 2$, $(E, \alpha_t, \nabla_t^\perp)$ will satisfy the Gauss equation

$$\langle R(X, Y)Z, W \rangle = \langle \alpha_t(X, W), \alpha_t(Y, Z) \rangle - \langle \alpha_t(X, Z), \alpha_t(Y, W) \rangle$$

the Codazzi equation

$$\begin{aligned} \nabla_{tX}^\perp \alpha(Y, Z) - \alpha_t(\nabla_X Y, Z) - \alpha_t(Y, \nabla_X Z) \\ = \nabla_{tY}^\perp \alpha(X, Z) - \alpha_t(\nabla_Y X, Z) - \alpha_t(X, \nabla_Y Z) \end{aligned}$$

and the Ricci equation

$$\langle R_t^\perp(X, Y)\xi, \eta \rangle = \langle (A_\xi \circ A_\eta - A_\eta A_\xi)(X), Y \rangle$$

for all X, Y, Z, W in TU , ξ, η in the normal space and for $0 \leq t \leq 1$ in such way that if $\{f_t\}$ denotes the family of isometric immersions, we will have $f_0(U)$ lying in a hyperplane of \mathbb{R}^{n+2} . Since M is simply connected, and α_t will be globally defined on $TM \times TM$, the family $\{f_t\}$ is also globally defined.

In order to construct E and $\{\xi_1, \xi_2\}$ as above we will prove the lemmas below. Given $x \in M$, $T_x M = T_x M_1 \oplus \mathbb{R}$, we will always denote by Z the tangent vector belonging to \mathbb{R} .

(4.1) **Lemma.** *With the same hypothesis of the theorem, if a point $x \in M$, $\alpha(Z, Z)(x) \neq 0$, then there is an orthonormal frame $\{\xi_1, \xi_2\}$ in the normal space, such that $A_{\xi_1}(T_x M_1) \subseteq T_x M_1$, $A_{\xi_1}(Z) = 0$ and $\text{rank } A_{\xi_2} = 1$.*

Proof. It follows easily from Lemma (3.1)(b).

We want to observe that, when $\alpha(Z, Z)$ is nonnull in a neighborhood V of x , $\alpha(Z, Z)$ will define a differentiable normal vector field on V . Since

ξ_2 can be taken as $\alpha(Z, Z)$ normalized, because $\alpha(Z, Z)$ is orthogonal to ξ_1 , ξ_2 is also a differentiable normal vector field on V implying that ξ_1 is a differentiable normal vector field on V too. Because $\text{rank } A_{\xi_2} = 1$, the operator A_{ξ_2} has eigenvalues with constant multiplicity on V ; this implies, by $[N_0]$, that its eigenvectors are differentiable vector fields on V . Consequently, their orthogonal projections onto $T_x M_1$, are differentiable vector fields on V . Then, for each $x \in V$, we can take differentiably the subspace W of $T_x M_1$ given by the orthogonal projection of the kernel of A_{ξ_2} onto $T_x M_1$, i.e.,

$$W = \{Y \in T_x M_1 / A_{\xi_2}(Y) = 0\}.$$

Since $\text{rank } A_{\xi_2} = 1$, we have that W is at least $(n - 2)$ -dimensional.

(4.2) **Lemma.** *With the notations above we have*

- (i) if $Y \in W$, $\nabla_Y^\perp \xi_1 = \nabla_Y^\perp \xi_2 = 0$,
- (ii) if λ is an eigenvalue of A_{ξ_1} at x , λ is constant along the segment of the geodesic $\gamma(t) = \exp_x tZ$ contained in V .

Proof. If $\dim W = n - 1$, $A_{\xi_2}|_{T_x M_1}$ is identically null and then A_{ξ_1} and A_{ξ_2} are diagonalized by the same basis. This implies that the normal curvature \mathcal{R}^\perp is zero and the lemma is obvious.

Now suppose $\dim W = n - 2$. By Lemma (4.1) we have $\langle \alpha(Y, Z), \xi_1 \rangle = 0$ and $\langle \alpha(Y, Z), \xi_2 \rangle = 0$ whenever $Y \in W$. Denoting by X the vector generating $T_x M_1 \cap W^\perp$, we can apply the Codazzi equation to X, Y, Z and ξ_1 to get

$$\begin{aligned} & \langle \nabla_X^\perp \alpha(Y, Z), \xi_1 \rangle - \langle \alpha(\nabla_X Y, Z), \xi_1 \rangle - \langle \alpha(Y, \nabla_X Z), \xi_1 \rangle \\ & = \langle \nabla_Y^\perp \alpha(X, Z), \xi_1 \rangle - \langle \alpha(\nabla_Y X, Z), \xi_1 \rangle - \langle \alpha(X, \nabla_Y Z), \xi_1 \rangle. \end{aligned}$$

Observe that the left-hand side is equal to zero, since $\alpha(Y, Z) = 0$, $\nabla_X Y$ and $A_{\xi_1}(\nabla_X Y)$ belong to $T_x M_1$ and $\nabla_X Z = 0$. The same reasons together with $\langle \alpha(X, Z), \xi_1 \rangle = 0$, imply that the right-hand side is equal to

$$\langle \nabla_Y^\perp \alpha(X, Z), \xi_1 \rangle = -\langle \alpha(X, Z), \nabla_Y^\perp \xi_1 \rangle.$$

Now the assumption $\dim W = n - 2$ implies $\langle \alpha(X, Z), \xi_2 \rangle \neq 0$. In fact, since $\text{rank } A_{\xi_2} = 1$, we have $A_{\xi_2}(X) = cA_{\xi_2}(Z)$ for some $c \neq 0$. Then

$$\langle \alpha(X, Z), \xi_2 \rangle = \langle A_{\xi_2} X, Z \rangle = c \langle A_{\xi_2} Z, Z \rangle = c \|\alpha(Z, Z)\| \neq 0.$$

Therefore $\langle \nabla_Y^\perp \xi_1, \xi_2 \rangle = 0$ implying $\nabla_Y^\perp \xi_1 = 0$. Expanding $Y \langle \xi_1, \xi_2 \rangle$ we get $\nabla_Y^\perp \xi_2 = 0$.

To prove the second part of the lemma, let us consider the orthonormal basis $\{X_1, \dots, X_{n-1}\}$ of $T_x M_1$ diagonalizing $A_{\xi_1}|_{T_x M_1}$. From the Codazzi equation applied to Z, X_i and ξ_1 we have

$$\begin{aligned} & Z \langle \alpha(X_i, X_i), \xi_1 \rangle - \langle \alpha(X_i, X_i), \nabla_Z^\perp \xi_1 \rangle - 2(\langle \alpha(\nabla_Z X_i, X_i), \xi_1 \rangle) \\ (4.3) \quad & = X_i \langle \alpha(Z, X_i), \xi_1 \rangle - \langle \alpha(Z, X_i), \nabla_{X_i}^\perp \xi_1 \rangle \\ & \quad - \langle \alpha(\nabla_{X_i} Z, X_i), \xi_1 \rangle - \langle \alpha(Z, \nabla_{X_i} X_i), \xi_1 \rangle. \end{aligned}$$

Now, we show that some terms in the above equation are null and we calculate the others. Since $A_{\xi_2}|_W = 0$ and $\nabla_Z^\perp \xi_1$ is parallel to ξ_2 , writing X_i as a linear combination of X and vectors belonging to a basis of W , we get

$$\langle \alpha(X_i, X_i), \nabla_Z^\perp \xi_1 \rangle = \langle X_i, X \rangle^2 \langle \alpha(X, X), \nabla_Z^\perp \xi_1 \rangle.$$

Because X_i is an eigenvector of A_{ξ_1} we have

$$\langle \alpha(\nabla_Z X_i, X_i), \xi_1 \rangle = \langle \nabla_Z X_i, X_i \rangle \langle \alpha(X_i, X_i), \xi_1 \rangle.$$

But $\langle \nabla_Z X_i, X_i \rangle = 0$ since X_i is of unitary length. On the right-hand side, the first and the last terms are clearly null since $A_{\xi_1}(Z) = 0$. The third one is also null because X_i is an eigenvector of A_{ξ_1} and then

$$\langle \alpha(\nabla_{X_i} Z, X_i), \xi_1 \rangle = \langle \nabla_{X_i} Z, X_i \rangle \langle \alpha(X_i, X_i), \xi_1 \rangle.$$

But $\langle \nabla_{X_i} Z, X_i \rangle = 0$ since M is a product and $X_i \in T_x M_1$. Now, we calculate the second term writing again X_i as a linear combination of X and vectors belonging to a basis of W . Using the facts that $A_{\xi_2}|_W = 0$ and $\nabla_Y^\perp \xi_2 = 0$, we have

$$\langle \alpha(Z, X_i), \nabla_{X_i}^\perp \xi_1 \rangle = \langle X_i, X \rangle^2 \langle \alpha(Z, X), \nabla_X^\perp \xi_1 \rangle.$$

Therefore, (4.3) is equivalent to

$$(4.4) \quad \begin{aligned} Z \langle \alpha(X_i, X_i), \xi_1 \rangle - \langle X_i, X \rangle^2 \langle \alpha(X, X), \nabla_Z^\perp \xi_1 \rangle \\ = -\langle X_i, X \rangle^2 \langle \alpha(Z, X), \nabla_X^\perp \xi_1 \rangle. \end{aligned}$$

Therefore, to prove (ii) of the lemma is equivalent to proving that

$$\langle \alpha(X, X), \nabla_Z^\perp \xi_1 \rangle = \langle \alpha(Z, X), \nabla_X^\perp \xi_1 \rangle,$$

which is equivalent to

$$(4.5) \quad \langle \alpha(X, X), \xi_2 \rangle \langle \nabla_Z^\perp \xi_1, \xi_2 \rangle = \langle \alpha(Z, X), \xi_2 \rangle \langle \nabla_X^\perp \xi_1, \xi_2 \rangle.$$

For that, consider the orthonormal basis $\{Z_1, Z_2, \dots, Z_n\}$ diagonalizing the operator A_{ξ_2} such that $A_{\xi_2}(Z_i) = 0$, $i \geq 2$. We can consider $Z_i \in W \subset T_x M_1$, for all $i \geq 3$. Applying the Codazzi equation to Z_2, Z_i ($i \geq 3$) and ξ_2 , we have

$$\begin{aligned} \langle \nabla_{Z_2}^\perp \alpha(Z_i, Z_i), \xi_2 \rangle - 2 \langle \alpha(\nabla_{Z_2} Z_i, Z_i), \xi_2 \rangle \\ = \langle \nabla_{Z_i}^\perp \alpha(Z_2, Z_i), \xi_2 \rangle - \langle \alpha(\nabla_{Z_i} Z_2, Z_i), \xi_2 \rangle - \langle \alpha(Z_2, \nabla_{Z_i} Z_i), \xi_2 \rangle. \end{aligned}$$

On the left-hand side, because $A_{\xi_2} Z_i = 0$, we will have only

$$(4.6) \quad \langle \nabla_{Z_2}^\perp \alpha(Z_i, Z_i), \xi_2 \rangle = -\langle \alpha(Z_i, Z_i), \nabla_{Z_2}^\perp \xi_2 \rangle.$$

Since $i \geq 3$ and $Z_i \in W$, from (i) of this lemma we have $\nabla_{Z_i}^\perp \xi_2 = 0$. Again, because $A_{\xi_2} Z_2 = A_{\xi_2} Z_i = 0$, the right-hand side is equal to zero. Then (4.5) becomes

$$\langle \alpha(Z_i, Z_i), \xi_1 \rangle \langle \nabla_{Z_2}^\perp \xi_2, \xi_1 \rangle = 0.$$

If there is i such that $\langle \alpha(Z_i, Z_i), \xi_1 \rangle \neq 0$, we have $\langle \nabla_{Z_2}^\perp \xi_2, \xi_1 \rangle = 0$. If $\langle \alpha(Z_i, Z_i), \xi_1 \rangle = 0$ for all $i \geq 3$, we have $A_{\xi_1}|_W = 0$ and then X is an eigenvector of A_{ξ_1} . Writing the Codazzi equation for Z_2, X, Z and ξ_1 we will have

$$\begin{aligned} \langle \nabla_{Z_2}^\perp \alpha(X, Z), \xi_1 \rangle - \langle \alpha(\nabla_{Z_2} X, Z), \xi_1 \rangle - \langle \alpha(X, \nabla_{Z_2} Z), \xi_1 \rangle \\ = \langle \nabla_X^\perp \alpha(Z_2, Z), \xi_1 \rangle - \langle \alpha(\nabla_X Z_2, Z), \xi_1 \rangle - \langle \alpha(Z_2, \nabla_X Z), \xi_1 \rangle. \end{aligned}$$

Since X is the only eigenvector of A_{ξ_1} with nonnull eigenvalue, $Z_2 \in \text{span}\{X, Z\}$ and $\langle \nabla_X X, Z \rangle = 0$, the only term we will have is

$$\langle \nabla_{Z_2}^\perp \alpha(X, Z), \xi_1 \rangle = -\langle \alpha(X, Z), \nabla_{Z_2}^\perp \xi_1 \rangle$$

which will be zero too. Since we are considering $\langle \alpha(X, Z), \xi_2 \rangle \neq 0$ (otherwise $\dim W = n - 1$) we have again $\nabla_{Z_2}^\perp \xi_1 = \nabla_{Z_2}^\perp \xi_2 = 0$. This fact will imply (4.4). In fact, write $Z_2 = aX + bZ$,

$$\begin{aligned} A_{\xi_2}(Z_2) = 0 &\Rightarrow a\langle \alpha(X, X), \xi_2 \rangle = -b\langle \alpha(X, Z), \xi_2 \rangle, \\ \nabla_{Z_2}^\perp \xi_1 = 0 &\Rightarrow a\nabla_X^\perp \xi_1 = -b\nabla_Z^\perp \xi_1. \end{aligned}$$

Therefore

$$(4.7) \quad b\langle \alpha(X, Z), \xi_2 \rangle \nabla_X^\perp \xi_1 = b\langle \alpha(X, X), \xi_2 \rangle \nabla_Z^\perp \xi_1.$$

Since we are supposing $b \neq 0$ [$(b = 0 \Rightarrow Z_2 \in T_x M_1 \Rightarrow \dim W = n - 1)$], (4.7) implies (4.5) taking product with ξ_2 .

(4.8) **Lemma.** *With the notation above, if $Y \in W$ and X is orthogonal to W , Y and X are parallel along the segment of geodesic $\gamma(t) = \exp_x tZ$ contained in V .*

Proof. If M_1 is two dimensional we consider isothermal parameters (x_1, x_2) defined on V such that $\partial/\partial x_1 = \lambda_1 X$ and $\partial/\partial x_2 = \lambda_2 Y$. If M_1 is at least three dimensional then $\dim W \geq 2$. Consider $Y_1, Y_2 \in W$ and the Codazzi equation for Y_1, Y_2, X and ξ_2 . We have

$$\begin{aligned} \langle \nabla_{Y_1}^\perp \alpha(Y_2, X), \xi_2 \rangle - \langle \alpha(\nabla_{Y_1} Y_2, X), \xi_2 \rangle - \langle \alpha(Y_2, \nabla_{Y_1} X), \xi_2 \rangle \\ = \langle \nabla_{Y_2}^\perp \alpha(Y_1, X), \xi_2 \rangle - \langle \alpha(\nabla_{Y_2} Y_1, X), \xi_2 \rangle - \langle \alpha(Y_1, \nabla_{Y_2} X), \xi_2 \rangle. \end{aligned}$$

Since $A_{\xi_2}(Y_i) = 0$ and $\nabla_{Y_i}^\perp \xi_2 = 0$ for $i = 1, 2$, we will have

$$\langle \nabla_{Y_1} Y_2, X \rangle \langle \alpha(X, X), \xi_2 \rangle = \langle \nabla_{Y_2} Y_1, X \rangle \langle \alpha(X, X), \xi_2 \rangle.$$

Since $\langle \alpha(X, X), \xi_2 \rangle \neq 0$ (otherwise $\dim W = n - 1$ which implies $\mathcal{R}^\perp = 0$ and then the lemma is obvious) we have $\langle [Y_1, Y_2], X \rangle = 0$ and then W is integrable. By Frobenius we can consider local coordinates (x_1, \dots, x_{n-1}, t) defined on V such that $\partial/\partial x_1 = \lambda_1 X$, $\partial/\partial x_i = \lambda_i Y_i$ for $Y_i \in W$ and $\partial/\partial t = \lambda_n Z$. Since $[\partial/\partial x_i, \partial/\partial t] = 0$, we have

$$\nabla_{\partial/\partial x_i} \frac{\partial}{\partial t} - \nabla_{\partial/\partial t} \frac{\partial}{\partial x_i} = \nabla_{\lambda_i Y_i} \lambda_n Z - \nabla_{\lambda_n Z} \lambda_i Y_i = 0.$$

Therefore

$$\lambda_i Y_i (\lambda_n Z) + \lambda_i \lambda_n \nabla_{Y_i} Z - \lambda_n Z (\lambda_i Y_i) - \lambda_n \lambda_i \nabla_Z Y_i = 0.$$

Since $\nabla_{Y_i} Z = 0$ (because M is a product) and $\nabla_Z Y_i$ is orthogonal to Z and Y_i , it follows that $\nabla_Z Y_i = 0$. The same proof implies $\nabla_Z X = 0$.

Now, we can start the proof of Theorem 2. Given $x \in M$, we consider the geodesic $\gamma(t) = \exp_x tZ$. We want to prove that for each point $p \in \gamma$, there is a neighborhood U of p such that we have differentiable normal vector fields ξ_1 and ξ_2 satisfying Lemmas (4.1) ($\text{rank } A_{\xi_2}$ does not need to be 1 but 0 or 1), (4.2) and (4.3).

First, we claim that there is a point $y \in \gamma$ such that $\alpha(Z, Z)(y) \neq 0$ because if not, f would take a complete geodesic to a straight line \mathbb{R}^{n+2} contradicting our assumption. As we have already observed, in a neighborhood U of y , we get ξ_1 and ξ_2 with the desired properties.

If $\alpha(Z, Z) = 0$ at an isolated point of γ , denoted by $\gamma(t_0)$, we have ξ_1 and ξ_2 defined on its right and $\bar{\xi}_1$ and $\bar{\xi}_2$ defined on its left. We claim that $\xi_1 = \bar{\xi}_1$ and $\xi_2 = \bar{\xi}_2$ (defined by limit) at the point $\gamma(t_0)$. In fact, since for each point $p \in \gamma$ there is a two-plane $\sigma \subset T_p M_1$ such that $k(\sigma) > 0$, for t close to t_0 and $t > t_0$, there exists $Y \in T_{\gamma(t)} M_1$ such that ξ_1 is $\alpha(Y, Y)$ normalized. For t close to t_0 and $t < t_0$, there also exists $\bar{Y} \in T_{\gamma(t)} M_1$ such that $\bar{\xi}_1$ is $\alpha(\bar{Y}, \bar{Y})$ normalized. Lemma (4.2)(ii) implies they are defined of the same manner, by limit, at $\gamma(t_0)$. Now, we define vector Y for $t \leq t_0$ and \bar{Y} for $t \geq t_0$ by parallel translation along the geodesic γ (see Lemma (4.8)). Since $\text{rank } A_{\xi_2} \leq 1$ and $\text{rank } A_{\bar{\xi}_2} \leq 1$, the curvature operator \mathcal{R} at $\gamma(t_0)$ is given by

$$\mathcal{R} = A_{\bar{\xi}_1} \wedge A_{\bar{\xi}_1} = A_{\xi_1} \wedge A_{\xi_1}.$$

Writing $\bar{\xi}_1$ as a linear combination of ξ_1 and ξ_2 , a straightforward calculation for one nonnull sectional curvature implies $\xi_1(t_0) = \pm \bar{\xi}_1(t_0)$. Therefore $\xi_1 = \bar{\xi}_1$ at $\gamma(t_0)$ otherwise Y and \bar{Y} would span a two-plane whose sectional curvature would be negative. Then

$$\xi_1 = \bar{\xi}_1 = \frac{\alpha(Y, Y)}{\|\alpha(\bar{Y}, \bar{Y})\|} = \frac{\alpha(\bar{Y}, \bar{Y})}{\|\alpha(\bar{Y}, \bar{Y})\|}$$

and this also proves the differentiability of ξ_1 in a neighborhood of $\gamma(t_0)$.

If $\alpha(Z, Z) = 0$ in a whole interval denoted by $[t_1, t_2]$, $\xi_1(t_1)$, $\xi_2(t_1)$, $\bar{\xi}_1(t_2)$ and $\bar{\xi}_2(t_2)$ are defined by limit since $\gamma(t_1)$ and $\gamma(t_2)$ are on the boundaries of neighborhoods where $\alpha(Z, Z) \neq 0$. We extend them to the interior of the interval by parallel translation along γ . First, we show that they have the desired properties. Consider X_i an eigenvector of A_{ξ_j} , $j = 1, 2$. From the Codazzi equation we have

$$\begin{aligned} & Z \langle \alpha(X_i, X_i), \xi_j \rangle - \langle \alpha(X_i, X_i), \nabla_Z^{\perp} \xi_j \rangle - 2 \langle \alpha(\nabla_Z X_i, X_i), \xi_j \rangle \\ &= X_i \langle \alpha(Z, X_i), \xi_j \rangle - \langle \alpha(Z, X_i), \nabla_{X_i}^{\perp} \xi_j \rangle - \langle \alpha(\nabla_{X_i} Z, X_i), \xi_j \rangle \\ &\quad - \langle \alpha(Z, \nabla_{X_i} X_i), \xi_j \rangle. \end{aligned}$$

Since $A_{\xi_i}(Z) = 0$, $\nabla_Z^{\perp} \xi_j = 0$ and $\nabla_{X_i} Z = 0$ we have

$$Z \langle \alpha(X_i, X_i), \xi_j \rangle = 0.$$

Now, repeating the same argument used above we show that the extensions agree on the interior of the interval. $\xi_1(t_1)$ is given by $\alpha(Y, Y)$ normalized, where $Y \in T_{\gamma(t_1)} M_1$ and $\xi_1(t_2)$ is given by $\alpha(\bar{Y}, \bar{Y})$ normalized, where $\bar{Y} \in T_{\gamma(t_2)} M_1$. Because on the interior of the interval $\xi_2(t)$ and $\bar{\xi}_2(t)$ have the desired properties, ξ_1 must be given by $\alpha(Y, Y)$ normalized and $\bar{\xi}_1$ by $\alpha(\bar{Y}, \bar{Y})$ normalized, $\text{rank } A_{\xi_2} \leq 1$ and $\text{rank } A_{\bar{\xi}_2} \leq 1$. Then $\xi_1(t) = \bar{\xi}_1(t)$, otherwise Y and \bar{Y} would span a plane whose sectional curvature would be negative.

To finish the proof of (ii) in Theorem 2, we have to prove that α_t defined previously, satisfies Gauss, Codazzi and Ricci equations. The Gauss equation will follow from (3.2). The Codazzi and Ricci equations follow easily if $0 < t \leq 1$. For $t = 0$, $\alpha_t(X, Y) = \langle \alpha(X, Y), \xi_1 \rangle \xi_1$, which implies that f_0 will be an isometric immersion in a hyperplane of \mathbb{R}^{n+2} . Therefore, we need to prove

only the Codazzi equation. This will follow from the Codazzi equation for the given isometric immersion f , remembering that for $Y \in W$,

$$\begin{aligned} \langle \nabla_Y^\perp \alpha(X_1, X_2), \xi_1 \rangle &= Y \langle \alpha(X_1, X_2), \xi_1 \rangle \quad \text{since } \nabla_Y^\perp \xi_1 = 0, \\ \langle \nabla_{X_1}^\perp \alpha(Y, X_2), \xi_1 \rangle &= X_1 \langle \alpha(Y, X_2), \xi_1 \rangle \quad \text{since } A_{\xi_2}(Y) = 0, \end{aligned}$$

for each X_1, X_2 tangent to M . Hence, we are reduced to prove them for X and Z , X orthogonal to W . The first of them is

$$\begin{aligned} X \langle \alpha(Z, Z), \xi_1 \rangle - 2 \langle \alpha(\nabla_X Z, Z), \xi_1 \rangle &= 0 \\ &= Z \langle \alpha(X, Z), \xi_1 \rangle - \langle \alpha(\nabla_Z X, Z), \xi_1 \rangle - \langle \alpha(X, \nabla_Z Z), \xi_1 \rangle \end{aligned}$$

because $A_{\xi_1}(Z) = 0$ and $\nabla_Z Z = 0$. The other equation will be

$$\begin{aligned} Z \langle \alpha(X, X), \xi_1 \rangle - 2 \langle \alpha(\nabla_Z X, X), \xi_1 \rangle &= Z \langle \alpha(X, X), \xi_1 \rangle \\ &= X \langle \alpha(Z, X), \xi_1 \rangle - \langle \alpha(\nabla_X Z, X), \xi_1 \rangle - \langle \alpha(Z, \nabla_X X), \xi_1 \rangle = 0 \end{aligned}$$

because $A_{\xi_1}(Z) = 0$ and $\nabla_X Z = \nabla_Z X = 0$. We can see that the only term left, $Z \langle \alpha(X, X), \xi_1 \rangle$ is zero from the Codazzi equation for f which has only one more term on the left-hand side, i.e., $\langle \alpha(X, X), \nabla_Z^\perp \xi_1 \rangle$ and only one more term on the right-hand side $\langle \alpha(Z, X), \nabla_Z^\perp \xi_1 \rangle$ which are equal by (4.4).

Now, part (i) of the theorem is easily proved. For each $\bar{x} \in M_1$, we consider $x = (\bar{x}, t) \in M_1 \times \mathbb{R}$, such that $\alpha(Z, Z)(x) \neq 0$. For a neighborhood U of x , which is $\bar{U} \times I$ (\bar{U} is a neighborhood of \bar{x} in M_1 and I an interval of \mathbb{R}) we have a differentiable normal vector ξ_1 , such that A_{ξ_1} satisfies the Gauss and Codazzi equations for an isometric immersion from \bar{U} to \mathbb{R}^n . Since M_1 is simply connected, we have the required immersion g .

We observe that f_0 , constructed above, is congruent to $g \times i$ since $A_{\xi_1}(Z) = 0$.

5. THEOREM 3 AND IMMERSIONS WHICH ARE NOT PRODUCTS

To prove Theorem 3, by Theorem 1 we have to consider only two cases, namely, when the soul is the circle S^1 and when $n = 3$ and the soul is homeomorphic to $\mathbb{R}P^2$. In both cases, the universal covering \tilde{M} of the manifold M will be the product $M_1 \times \mathbb{R}$.

We will denote by $\tilde{f} = f \circ P$ (P is the covering map), $\tilde{\alpha}$ the second fundamental form of \tilde{f} , and $\nu_{\tilde{f}}(\tilde{x})$ the index of relative nullity of \tilde{f} at $\tilde{x} \in \tilde{M}$. Let us take $\tilde{x} \in \tilde{M}$ such that $\nu_{\tilde{f}}(\tilde{x}) = 0$. Following the proof of Lemma (3.1) we see that at a point x where $\nu(x) = 0$, there are normal vectors ξ_1 and ξ_2 such that $A_{\xi_1}(T_x M_1) = T_x M_1$, $A_{\xi_1}(Z) = 0$ and $\text{rank } A_{\xi_2} = 1$ (Z denotes the tangent vector belonging to \mathbb{R}). Since $\mathcal{R}^\perp = 0$, the same orthonormal basis diagonalizes A_{ξ_1} and A_{ξ_2} . Then Z_1 , eigenvector of A_{ξ_2} , corresponding to nonnull eigenvalue, has to be tangent either to M_1 or to \mathbb{R} . If $Z_1 \in T_x M_1$, $A_{\xi_2}(Z) = 0$ and this implies that Z is relative nullity vector, which contradicts $\nu_{\tilde{f}}(\tilde{x}) = 0$. Then $Z_1 = Z$ and $\tilde{\alpha}(X, Z) = 0$ for all $X \in T_x M_1$. The proof of Theorem 2 in $[M_1]$ implies $\tilde{\alpha}(X, Z) = 0$ for all $X \in TM_1$ since no complete geodesic in \tilde{M} can be taken to a straight line in \mathbb{R}^{n+2} . Therefore f is reducible along the soul A . By Proposition (2.5) M is a product over the soul and because $\alpha(X, Z) = 0$, f is a product of hypersurface immersions, proving the theorem.

Now, we show an example of a complete manifold M^n with S^1 as a soul, isometrically immersed in \mathbb{R}^{n+2} which is not a product of immersions. The reader is referred to [A-M, §4], for the details.

Let $f_1: M_1^{n-1} \rightarrow \mathbb{R}^n$ be any isometric immersion of codimension one and $f_2: S^1 \rightarrow \mathbb{R}^2$ an isometric immersion such that f_2 is totally geodesic on the interval $I = (-\varepsilon, \varepsilon)$. Now consider $f = f_1 \times f_2: M = M_1 \times S^1 \rightarrow \mathbb{R}^{n+2}$. The restriction of the immersion f to $M_1 \times I$ is given by

$$(m, t) \mapsto \sum_{i=1}^{n+1} f_1^i(m)e_i + te_{n+2}$$

where $\{e_1, \dots, e_{n+2}\}$ is the standard basis of \mathbb{R}^{n+2} .

If M_1 is compact in §4 of [A-M], the authors show how to construct an isometric immersion f of $M_1 \times I$ which is not a product of hypersurface immersions and agrees with f whenever $\varepsilon/2 < |t| < \varepsilon$.

We want to point out that the compactness of M_1 is necessary to guarantee that equation (7) in [A-M] never vanishes and then h_1 defined by them is an immersion. We are interested when M_1 is noncompact. So, we will fix one copy S^1 of M , denote it by \bar{S} , and for a positive real number σ , take the compact subset $B(\bar{S}, \sigma) = \{m \in M/d(m, \bar{S}) < \sigma\}$ ($d(m, \bar{S})$ is the distance from m to \bar{S}). We redefine the immersion $h: M_1 \times I \rightarrow \mathbb{R}^{n+2}$ by

$$h(m, t) = \sum_{i=1}^{n-1} f_1^i(m)e_i + f_1^n(m)[X_n + \alpha X_{n+1}](t) + \gamma(t)$$

where now α is a smooth function depending on (m, t) . For $m \in B(\bar{S})$, $\alpha(m, t)$ satisfies the same conditions as α in [A-M] as a function of t , and we require that $\alpha(m, t)$ goes to zero when m is such that $d(m, \bar{S}) = \sigma \cdot \gamma$ is as in [A-M].

Following the same procedure as in [A-M] we can produce an isometric immersion \bar{f} of $M_1 \times I$ which agrees with f whenever $\varepsilon/2 < |t| < \varepsilon$ and $m \notin B(\bar{S}, \sigma)$.

Piecing f and \bar{f} together we obtain a 2-codimensional isometric immersion of $M_1 \times S^1$ which is not a product of immersions.

Before finishing this paper we would like to say that it would be of interest to construct examples of manifolds having S^1 as soul in \mathbb{R}^{n+2} , which are not product over the soul.

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